Trace formulae and principal functions of Hilbert space operators(Recent Developments in Linear Operator Theory and its Applications)

Cho, Muneo

数理解析研究所講究録 1458 巻 2005 年 117-126

Trace formulae and principal functions of Hilbert space operators

長 宗雄 (Muneo Chō)
神奈川大学工学部
Department of Mathematics
Kanagawa University

This paper is the results of [13], [14] and [15]. Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. About the trace formula, we have the following:

Theorem 1 (M. Krein, 1953). *Let A be a self-adjoint operator on $\mathcal{H}$ and K be a trace class self-adjoint operator on $\mathcal{H}$. Then there exists a unique function $\delta(t)$ such that*

$$\mathrm{Tr}\Big(p(A+K)-p(A)\Big)=\int p'(t)\delta(t)dt,$$

where p is a polynomial.

Let $\mathcal{C}_1$ be the trace class and $\mathcal{A}$ be the set of all Laurent polynomials; $\mathcal{P}(r,z)=\sum_{k=-N}^{N}p_k(r)z^k$. Let $J(\mathcal{P},\mathcal{Q})$ be the Jacobian of $\mathcal{P}$, $\mathcal{Q}$.

Theorem 2 (Carey-Pincus [8], Helton-Howe [19]). *Let $T=X+iY$ be an operator on $\mathcal{H}$ with trace class self-commutator ($[T^*,T]\in\mathcal{C}_1$). Then there exists a function $g(x,y)$ such that*

$$\mathrm{Tr}\Big([p(X,Y),q(X,Y)]\Big)=\frac{1}{2\pi i}\int\int J(p,q)(x,y)g(x,y)dx\,dy,$$

where p and q are polynomials of two variables.

Functions $\delta(t)$ and $g(x,y)$ in Theorems 1 and 2 are called the phase shift of the perturbation problem $A\to A+K$, and the (Cartesian) principal function of T, respectively. Let T be hyponormal and satisfy $[T^*,T]\in\mathcal{C}_1$. For operators A and K of Theorem 1, let $A=TT^*$ and $K=T^*T-TT^*(=[T^*,T]\in\mathcal{C}_1)$. Then Theorem 1 is

$$\mathrm{Tr}\Big(p(T^*T)-p(TT^*)\Big)=\int p'(t)\delta(t)dt.$$

And

$$\delta(t) = \frac{1}{2\pi}\int_0^{2\pi} g(\sqrt{t}\cos\theta, \sqrt{t}\sin\theta)d\theta \quad \text{a.e. } t > 0.$$

Let $\mathcal{A}$ be the linear space of all Laurent polynomials $\mathcal{P}(r,z)$ with polynomial coefficients such that $\mathcal{P}(r,z) = \sum_{k=-N}^{N} p_k(r)z^k$, where N is a non-negative integer and every $p_k(r)$ is a polynomial of one variable. For $T = U|T|$ with unitary U, put $\mathcal{P}(|T|, U) = \sum_{k=-N}^{N} p_k(|T|)U^k$.

For the polar decomposition $T = U|T|$, we have the following:

Theorem 3 ([8],[11],[25]). *Let $T = U|T|$ be semi-hyponormal operator satisfying $[|T|, U] \in \mathcal{C}_1$ with unitary U. Then there exists a function g_T such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$\mathrm{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta} g_T(e^{i\theta}, r)drd\theta.$$

Definition 1. T is p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. Especially, T is called hyponormal and semi-hyponormal if $p = 1$ and $p = 1/2$, respectively. It holds

$$\text{hyponormal} \implies \text{semi-hyponormal} \implies p\text{-hyponormal}.$$

(1) If $T = U|T|$ is semi-hyponormal, then $S = U|T|^{\frac{1}{2}}$ is hyponormal.

(2) If $T = U|T|$ is semi-hyponormal with $1 \notin \sigma(U)$, then $R = L^{-1}(U) + i|T|$ is hyponormal, where $L^{-1}(U) = i(U+1)(U-1)^{-1}$.

(3) If $T = U|T|$ is p-hyponormal, then $T_t = |T|^t U |T|^{1-t}$ is q-hyponormal (Aluthge transformation), where $q = \min\{p+t, p+1-t, 1\}$. Hence, if T is semi-hyponormal, then T_t is hyponormal.

Spectral mapping theorem holds for these transformations; i.e.,
if $T = U|T|$ be semi-hyponormal, then
(1) $\sigma(S) = \{\sqrt{r}e^{i\theta} : re^{i\theta} \in \sigma(T)\}$;
(2) $\sigma(R) = \{L^{-1}(e^{i\theta}) + ir : re^{i\theta} \in \sigma(T)\}$;
(3) $\sigma(T_t) = \sigma(T)$.

Functions g and g_T of Theorems 2 and 3 are called *the principal functions* of T related to the Cartesian decomposition $T = X + iY$ and the polar decomposition $T = U|T|$, respectively. For a hyponormal operator $T = X + iY$, the principal function of T is defined by the mosaic $0 \Box B(x,y) \Box I$ as follows;

$$g(x,y) = \mathrm{Tr}(B(x,y)).$$

For a semi-hyponormal operator $T = U|T|$, the principal function of T is defined by the mosaic $0 \square B^P(e^{i\theta}, r) \square I$ as follows;

$$g_T(e^{i\theta}, r) = \mathrm{Tr}(B^P(e^{i\theta}, r)).$$

For an operator $T = U|T|$ let $T_t = |T|^t U |T|^{1-t}$ $(0 < t < 1)$ be the Aluthge transformation of T. Let g_T and g_{T_t} be the principal functions of T and $T_t = |T|^t U|T|^{1-t}$ $(0 < t < 1)$, respectively. Then we have following results ([12]):

(1) If T is invertible p-hyponormal with $[|T|, U] \in \mathcal{C}_1$, then $g_T = g_{T_t}$.

(2) If T is hyponormal with $[|T|, U] \in \mathcal{C}_1$, then $g(x, y) = g_T(e^{i\theta}, r)$ for $x + iy = re^{i\theta}$.

The following results are important.

Theorem 4 ([12]). *If a positive invertible operator A and an operator D satisfy $[A, D] \in \mathcal{C}_1$, then, for any real number α, we have*

$$[A^\alpha, D] \in \mathcal{C}_1.$$

Theorem 5 ([12]). *If T is an invertible operator such that $[T^*, T] \in \mathcal{C}_1$, then $[\tilde{T}^*, \tilde{T}] \in \mathcal{C}_1$, where $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ (Aluthge transform of T).*

For an invertible operator $T = U|T|$, $[T^*, T] \in \mathcal{C}_1$ if and only if $[|T|, U] \in \mathcal{C}_1$, because

$$[|T|^2, U] = [T^*, T]U \quad \text{and} \quad [T^*, T] = |T|[|T|, U]U^* + [|T|, U]|T|U^*.$$

1. Trace formula of p-hyponormal operators II

Let $\mathbf{T} = \{e^{i\theta} | 0 \square \theta < 2\pi\}, \Sigma$ be the set of all Borel sets in $\mathbf{T}$ and m be a measure on the measure space $(\mathbf{T}, \Sigma)$ such that $dm(\theta) = \frac{1}{2\pi} d\theta$. Then we have

Theorem 3'. *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of T. Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then there exists a summable function g_T such that, for $\mathcal{P}(r, z), \mathcal{Q}(r, z) \in \mathcal{A}$, it holds*

$$\mathrm{Tr}\Big([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)] \Big) = \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

We denote by $C^\infty(\mathbf{R})$ the set of all smooth functions on $\mathbf{R}$ and by $C_c^\infty(\mathbf{R})$ the set of all functions in $C^\infty(\mathbf{R})$ with compact support. We denote by $\mathcal{B}$ the linear space of all Laurent polynomials $\phi(r, z)$ such that $\phi(r, z) = \sum_{k=-N}^{N} f_k(r) z^k$, where every $f_k \in C^\infty(\mathbf{R})$. For $T = U|T|$ with unitary U, put $\phi(|T|, U) = \sum_{k=-N}^{N} f_k(|T|) U^k$ for $\phi \in \mathcal{B}$.

In [6], Carey and Pincus proved a more general version of Theorem 3. It requires complicate calculations. Using polynomial approximation, we improve Theorem 3 in the following form.

Theorem 1.1. *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of T. Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $\phi, \psi \in \mathcal{B}$, it holds*

$$\operatorname{Tr}\Big([\phi(|T|, U), \psi(|T|, U)] \Big) = \int \int J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

For the proof of Theorem 1.1, it needs Theorems 1.2 - 1.5.

Theorem 1.2. *Let $A, \{B_j\}_{j=1,\cdots,n}$ be operators such that $[A, B_j] \in \mathcal{C}_1$ and $||B_j|| \square r$ for all j $(j = 1, 2, ..., n)$. Then*

$$||[A, B_1 B_2 \cdots B_n]||_1 \square n r^{n-1} \max_j ||[A, B_j]||_1.$$

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T. Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then we have

$$[U, e^{it|T|}] = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [U, |T|^n].$$

By Theorem 1.2,

$$||[U, e^{it|T|}]||_1 \square \sum_{n=1}^{\infty} \frac{|t|^n n ||T||^{n-1}}{n!} ||[U, |T|]||_1 \square |t| \cdot ||[U, |T|]||_1 \, e^{|t| \cdot ||T||}.$$

Definition 1.1. Under the assumption above, we define a constant c_T of an operator $T = U|T|$ satisfying $[U, |T|] \in \mathcal{C}_1$ by

$$c_T = \max_{|t| \square 1} ||[U, e^{it|T|}]||_1.$$

Proof of the next theorem is based on an idea of the proof of [19, Lemma 3.2].

Theorem 1.3. *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T. Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $f \in \mathcal{S}$ and an integer n, it holds*

$$||[U^n, f(|T|)]||_1 \square |n| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_T(|t| + 1) |\hat{f}(t)| dt,$$

where c_T is the constant of Definition 1.1.

Theorem 1.4. *Let F be a compact set of $\mathbf{R}$ and $f \in C^{\infty}(\mathbf{R})$. Then there exist a function $f_1 \in C_c^{\infty}(\mathbf{R})$, a sequence $\{p_n\}$ of polynomials and a sequence $\{\gamma_n\}$ in $C_c^{\infty}(\mathbf{R})$ such that*

$$f(x) = f_1(x), \quad p_n(x) = \gamma_n(x) \ \ for \ \ x \in F,$$

$$\sup_{y\in F}|f_1(y)-\gamma_n(y)|\to 0 \qquad (n\to\infty),$$

$$\sup_{y\in F}|f_1^{(1)}(y)-\gamma_n^{(1)}(y)|\to 0 \qquad (n\to\infty),$$

$$\sup_{t\in\mathbf{R}}|\hat{f}_1(t)-\hat{\gamma}_n(t)|\to 0 \qquad (n\to\infty),$$

and

$$\sup_{t\in\mathbf{R}}|t|^3\cdot|\hat{f}_1(t)-\hat{\gamma}_n(t)|\to 0 \qquad (n\to\infty).$$

Theorem 1.5. *Let $T=U|T|$ be the polar decomposition. Assume that U is unitary and $[U,|T|]\in\mathcal{C}_1$. Then, for $f,g\in\mathcal{S}$ and integers m,n, it holds*

$$\begin{aligned}&|||[f(|T|)U^n, g(|T|)U^m]|||\\ \square\ &|n|\cdot|||f|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{g}(t)|dt\\ +&|m|\cdot|||g|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{f}(t)|dt\end{aligned}$$

where c_T is the constant of Definition 1.1 *and* $|||h|||=\sup_{x\in\sigma(|T|)}|h(x)|$.

Next we apply this result to p-hyponormal operators $(0<p<1/2)$.

Definition 1.2. Let $T=U|T|$ be p-hyponormal with unitary U. Put $S=U|T|^{2p}$. Then S is semi-hyponormal with unitary U. Hence there exists the Pincus principal function g_S of S and we define the principal function g_T of T by

$$g_T(e^{i\theta},r)=g_S(e^{i\theta},r^{\frac{1}{2p}})$$

(see [11, Definition 3]).

The following theorem is a generalization of Theorem 10 of [11].

Theorem 1.6. *Let $T=U|T|$ be an invertible p-hyponormal operator. If $|T|^{2p}-U|T|^{2p}U^*\in\mathcal{C}_1$, then for $\mathcal{P}(r,z),\mathcal{Q}(r,z)\in\mathcal{A}$ it holds*

$$\mathrm{Tr}\Big([\mathcal{P}(|T|,U),\mathcal{Q}(|T|,U)]\Big)=\int\int J(\mathcal{P},\mathcal{Q})(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)drdm(\theta).$$

2. Trace formulae associated with the polar decomposition of operators

Let $\mathcal{S}(\mathbf{R}^2)$ be the Schwartz space of rapidly decreasing functions at infinity. For $T=X+iY$, let $\mathcal{E}$ and $\mathcal{F}$ be the spectral measures of self-adjoint operators X and Y, respectively. We define τ on $\mathcal{S}(\mathbf{R}^2)$ by

$$(*)\qquad \tau(\phi)=\int\int\phi(x,y)d\mathcal{E}(x)d\mathcal{F}(y)\quad(\phi\in\mathcal{S}(\mathbf{R}^2)).$$

By a standard argument, we have

$$\int\int e^{itX}e^{isY}\hat{\phi}(t,s)dtds = \int\int \phi(x,y)d\mathcal{E}(x)d\mathcal{F}(y),$$

where

$$\hat{\phi}(t,s) = \frac{1}{2\pi}\int\int e^{-i(tx+sy)}\phi(x,y)dxdy$$

is the Fourier transform of the function ϕ (see, for example, [22, p.237]).

Put $\nu(E) = \int\int_E \hat{\phi}(t,s)dtds$ for a measurable set $E \subset \mathbf{R}^2$. Since $\hat{\phi}(t,s) \in \mathcal{S}(\mathbf{R}^2)$, we have

$$\int\int (1+|t|)(1+|s|)|\hat{\phi}(t,s)|dtds < \infty.$$

Following Carey-Pincus [8], put $G(x,y) = \int\int e^{itx+isy}d\nu(t,s)$ and define

$$G(X,Y) = \int\int G(x,y)d\mathcal{E}(x)d\mathcal{F}(y).$$

Then

$$\tau(\phi) = \int\int e^{itX}e^{isY}\nu(t,s)dtds = G(X,Y).$$

Note here that we have $\tau(\psi) = \tau(\phi)$ for any smooth function $\psi(x,y)$ which coincides with $\phi(x,y)$ on $\mathrm{supp}(\tau)$.

The map $\tau : \mathcal{S}(\mathbf{R}^2) \to B(\mathcal{H})$ has the following properties [22, Chapter X, §2];

(1) τ is linear, continuous and $\mathrm{supp}\,(\tau) \subseteq \sigma(X)\times\sigma(Y)$,

(2) $\tau(1) = I, \tau(p+q) = p(X)+q(Y)$ for polynomials p and q of one varaible of x and y, respectively.

(3) $\tau(\phi)\tau(\psi) - \tau(\phi\psi) \in \mathcal{C}_1$ for $\phi,\psi \in \mathcal{S}(\mathbf{R}^2)$,

(4) $\tau(\phi)^* - \tau(\bar{\phi}) \in \mathcal{C}_1$.

By (3) we have an important property $[\tau(\phi),\tau(\psi)] \in \mathcal{C}_1$ for $\phi,\psi \in \mathcal{S}(\mathbf{R}^2)$.

The following theorem is a basis.

Theorem 6 (Carey-Pincus, [8, Theorem 5.1]**).** *Let $T = X+iY$ be an operator with $[T^*,T] \in \mathcal{C}_1$. Let $\mathcal{E},\mathcal{F}$ be the spectral measures of X and Y, respectively and τ be given by $(*)$. Then there exists a summable function g such that, for $\phi,\psi \in \mathcal{S}(\mathbf{R}^2)$,*

$$\mathrm{Tr}\Big([\tau(\phi),\tau(\psi)]\Big) = \frac{1}{2\pi i}\int\int J(\phi,\psi)(x,y)g(x,y)dxdy.$$

Moreover, if T is hyponormal, then $g \geq 0$ and $g(x,y) = 0$ for $x+iy \notin \sigma(T)$.

In this section, the main theorem is Theorem 2.7. We prepare some results.

Lemma 2.1. *Let A be a positive invertible operator and operators D, E, F satisfying $[A, D], [E, D], [F, D] \in \mathcal{C}_1$. Then for any real number α, we have*

$$[EA^{\alpha}F, D] \in \mathcal{C}_1.$$

Lemma 2.2 (cf.p.158 of [8]**).** *Let $T = X + iY$ be an invertible operator such that $[T^*, T] \in \mathcal{C}_1$. Let $\psi \in \mathcal{S}(\mathbf{R}^2)$, $D = \tau(\psi)$ and operators E, F satisfy $[E, D], [F, D] \in \mathcal{C}_1$. Then, for $\phi(x, y) = (x^2 + y^2)^{\alpha}$ with a real number α,*

$$\mathrm{Tr}\Big([E\tau(\phi)F, D]\Big) = \mathrm{Tr}\Big([E|T|^{2\alpha}F, D]\Big).$$

Theorem 2.3. *Let $T = U|T|$ be an invertible operator with $[T^*, T] \in \mathcal{C}_1$ and let g be the principal function assoicated with the Cartesian decomposition of $T = X + iY$. Then there exists a summable function g_T such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$\mathrm{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drd\theta,$$

and $g_T(e^{i\theta}, r) = g(x, y)$ almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.

An invertible operator T is said to be log-hyponormal if $\log T^*T \geq \log TT^*$ [17].

For the proof of the next result, we need the following two lemmas. For an operator T, let $\sigma_{ap}(T)$ and $\sigma_p(T)$ be the approximate point spectrum and the point spectrum of T, respectively. The following lemmas are important for the main theorem.

Lemma 2.4. *Let $T = U|T|$ be an invertible semi-hyponormal operator with $[|T|, U] \in \mathcal{C}_1$. Then the principal function g^P associated with the polar decomposition $T = U|T|$ of T satisfies $g_T(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(T)$.*

Lemma 2.5. *Let $T = U|T|$ be an operator with unitary U and put $S = U(|T| + I)$. If $z \in \partial\sigma(S)$, then $|z| \geq 1$. Therefore, if $z \in \sigma(S)$, then $|z| \geq 1$.*

By the above lemmas, for the next theorem we can give another proof of [11, Theorem 9].

Theorem 2.6. *Let $T = U|T|$ be a semi-hyponormal operator with unitary U and $[|T|, U] \in \mathcal{C}_1$. Then there exists a summable function g_T such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$\mathrm{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drd\theta.$$

Theorem 2.7. *Let $T = X + iY = U|T|$ be a semi-hyponormal operator with unitary U and $[|T|, U] \in \mathcal{C}_1$. If g and g_T are the principal function associated with the Cartesian decomposition of T and the summable function in Theorem 2.6, respectively, then*

$$g(x, y) = g_T(e^{i\theta}, r)$$

almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.

3. Principal functions for high powers of operators

In this section, we denote g_T and g_T^P be the principal functions of the Cartesian decomposition and the polar decomposition of T, respectively. C.A. Berger gave the principal functions g_{T^n} of powers T^n of T in terms of g_T and proved that for a sufficiently high n, T^n has a non-trivial invariant subspace for a hyponormal operator T ([5]). For a hyponormal operator T with $[T^*, T] \in \mathcal{C}_1$, it holds that

$$g_{T^n}(z) = \sum_{k=1}^{n} g_T(\zeta_k), \quad \text{where } \zeta_k^n = z \ (k = 1, ..., n).$$

More generally, in [4], for a polynomial p, it holds

$$g_{p(T)}(z) = \sum_{\zeta} \{ \ g_T(\zeta) \ : \ p(\zeta) = z \ \}.$$

Theorem 3.1. *Let $T = X + iY = U|T|$ be an operator satisfying the following trace formula:*

$$\mathrm{Tr}([\phi(|T|, U), \psi(|T|, U)]) = \frac{1}{2\pi} \int \int J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g_T^P(e^{i\theta}, r) dr d\theta$$

for any Laurent polynomials ϕ and ψ. Then the principal function $g_T(x, y)$ related to the Cartesian decomposition $T = X+iY$ of T exists and it is given by $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.

If an operator $T = U|T|$ is invertible and $[|T|, U] \in \mathcal{C}_1$, then $[T^*, T] \in \mathcal{C}_1$. So we have the following

Corollary 3.2. *If an invertible operator $T = X + iY = U|T|$ satisfies $[|T|, U] \in \mathcal{C}_1$, then $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.*

For a relation between g_T^P and $g_{T^n}^P$, we need the following Berger's result:

Theorem 3.3 (Berger, [5, Theorem 4]**).** *For an operator T, if $[T^*, T] \in \mathcal{C}_1$, then for a positive integer n,*

$$g_{T^n}(x, y) = \sum_{(u+iv)^n = x+iy} g_T(u, v).$$

Theorem 3.4. *For an operator T with $[T^*, T] \in \mathcal{C}_1$, if $\int \int g(x, y) dx dy \neq 0$, then*

$$\lim_{n \to \infty} \text{ess sup} \, |g_{T^n}| \ = \ \infty.$$

Applying Corollary 3.2 and Theorem 3.4 to T^n, we have the following.

Corollary 3.5. *For an operator $T = U|T|$, let $T^n = U_n|T^n|$ be the polar decomposition of T^n $(n = 1, 2, ...)$. If $[|T^n|, U_n] \in \mathcal{C}_1$ for every non-negative integer n and*

$\int\int g_T^P(e^{i\theta}, r) r d\theta dr \neq 0$, *then*

$$\lim_{n\to\infty} \text{ess sup}\, |g_{T^n}^P| = \infty.$$

Let $T = U|T|$ and $T^n = U_n|T^n|$ be the polar decompositions of T and T^n, respectively. Then it holds that if $[T^*, T] \in \mathcal{C}_1$, then $[T^{*n}, T^n] \in \mathcal{C}_1$ for any positive integer n. On the other hand, in the polar decomposition case, it is not clear whether $[|T|, U] \in \mathcal{C}_1$ impilies $[|T^n|, U_n] \in \mathcal{C}_1$ even if $n = 2$. If T is invertible and $[|T|, U] \in \mathcal{C}_1$, then, for every n, it holds $[|T^n|, U_n] \in \mathcal{C}_1$ by [11, Theorem 3].

Next we consider operators with cyclic vectors. First we need the following result.

Theorem 3.6 (Martin and Putinar, Th.X.4.3 [22]**).** *Let g_T and g_V be the principal functions of operators T and V such that $[T^*, T], [V^*, V] \in \mathcal{C}_1$, respectively. If there exists an operator $A \in \mathcal{C}_1$ such that $AV = TA$ and $\ker(A) = \ker(A^*) = \{0\}$, then $g_T \Box g_V$.*

Proof of the following lemma is based on it of [22, Corollary X.4.4].

Lemma 3.7. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If T has a cyclic vector, then $g_T \Box 1$.*

Let S be an operator having the principal function g_S related to the Cartesian decomposition $S = X + iY$. Then $g_{S^*}(x, y) = -g_S(x, -y)$. Hence, as a corollary of Lemma 3.7, we have the following.

Lemma 3.8. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If T^* has a cyclic vector, then $-1 \Box g_T$.*

Theorem 3.9. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If $\int\int g_T(x, y) dx dy \neq 0$, then, for a sufficiently high n, T^n has a non-trivial invariant subspace.*

REFERENCES

1. A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integr. Equat. Oper. Th. **13**(1990), 307-315.
2. A. Aluthge and D. Wang, w-hyponormal operators II, Integr. Equat. Oper. Th. **37**(2000), 324-331.
3. S.K. Berberian, Approximate proper vectors, Proc. Amer. Math. Soc. **13**(1062), 111-114.
4. C.A. Berger, Sufficiently high power of hyponormal operators have rationally invariant subspaces, Integr. Equat. Oper. Th. **1/3**(1978), 444-447.
5. C.A. Berger, Intertwined operators and the Pincus principal function, Integr. Equat. Oper. Th. **4**(1981), 1-9.
6. R.W. Carey and J.D. Pincus, Almost commuting algebras, Lecture Notes in Math., **574** Springer-Verlag, Belrin (1973), 19-43.

7. R.W. Carey and J.D. Pincus, An invariant for certain operator algebras, Proc. Nat. Acad. Sci. U.S.A. **71**(1974), 1952-1956.
8. R.W. Carey and J.D. Pincus, Mosaics, principal functions, and mean motion in von-Neumann algebras, Acta Math. **138**(1977), 153-218.
9. M. Chō and M. Itoh, Putnam's Inequality for p-hyponormal operators, Proc. Amer. Math. Soc. **123** (1995), 2435-2440.
10. M. Chō and T. Huruya, Aluthge transformations and invariant subspaces of p-hyponormal operators, Hokkaido Math. J. **32** (2003), 445-450.
11. M. Chō and T. Huruya, Trace formulae of p-hyponormal operators, Studia Math. **161**(2004), 1-18.
12. M. Chō and T. Huruya, Relation between principal functions of p-hyponormal operators, J. Math. Soc. Japan **57**(2005), 605-618.
13. M. Chō and T. Huruya, Trace formulae of p-hyponormal operatorsII, Hokkaido Math. J. to appear.
14. M. Chō, T. Huruya and C. Li, Trace formulae associated with the polar decompositions of operators, Math. Proc. Royal Irish Acad. to appear.
15. M. Chō, T. Huruya, C. Li and A-H. Kim, Principal functions for high powers of operators, Tokyo J. Math. to appear.
16. K.F. Clancey, Seminormal operators, Springer Verlag Lecture Notes No. 742, Berlin-Heidelberg-New York, 1979.
17. T. Furuta, Invitation to linear operators, Taylor & Francis Inc, London and New York, 2001.
18. I. C. Gorberg and M. G. Krein, Introduction to the theory of linear nonselfadjoint operators, Translation Math. Monographs, Vol. 18, Amer. Math. Soc., Providence, R.I., 1969.
19. J.W. Helton and R. Howe, Integral operators, commutator traces, index and homology, Proceedings of a conference on operator theory, Springer Verlag Lecture Notes No. 345, Berlin-Heidelberg-New York, 1973.
20. I.B. Jung, E. Ko and C. Pearcy, Aluthge transforms of operators, Integr. Equat. Oper. Th. **37**(2000), 437-448.
21. I.B. Jung, E. Ko and C. Pearcy, Spectral pictures of Aluthge transforms of operators, Integr. Equat. Oper. Th. **40**(2001), 52-60.
22. M. Martin and M. Putinar, Lectures on hyponormal operators, Birkhäuser Verlag, Basel, 1989.
23. J.D. Pincus and D. Xia, Mosaic and principal function of hyponormal and semi-hyponormal operators, Integr. Equat. Oper. Th. **4**(1981), 134-150.
24. K. Tanahashi, On log-hyponormal operators, Integr. Equat. Oper. Th. **34**(1999), 364-372.
25. D. Xia, Spectral theory of hyponormal operators, Birkhäuser Verlag, Basel, 1983.